Note that when π_{it} is estimated by (5.97), the asymptotic variance in (5.94) is likely to underestimate the variability of $\widehat{\boldsymbol{\theta}}$. In Chapter 4, we discussed how to account for the variability of the estimate $\widehat{\boldsymbol{\gamma}}$ in the model for $\pi_{it}(\boldsymbol{\gamma})$. If $\widehat{\boldsymbol{\gamma}}$ is obtained from GEE or maximum likelihood, we can readily find the asymptotic variance of $\widehat{\boldsymbol{\theta}}$ adjusted for the variability of $\widehat{\boldsymbol{\gamma}}$.

Consider a one-sample U-statistic vector with q arguments of the form $\mathbf{U}_n(\boldsymbol{\gamma})$, where $\boldsymbol{\gamma}$ is the parameter vector describing the missing data model as in the above example. It follows from the theory of multivariate statistics that

$$\sqrt{n}\left(\mathbf{U}_n(\boldsymbol{\gamma})-\boldsymbol{\theta}\right)=q\frac{\sqrt{n}}{n}\sum_{i=1}^{n}\widetilde{\mathbf{h}}_1\left(\mathbf{y}_i,\mathbf{r}_i,\boldsymbol{\gamma}\right)+\mathbf{o}_p(1) \tag{5.98}$$

As in Chapter 4, let $\widehat{\boldsymbol{\gamma}}$ be the solution to the estimating equation: $\mathbf{w}_n(\boldsymbol{\gamma})=\frac{1}{n}\sum_{i=1}^{n}\mathbf{w}_{ni}(\boldsymbol{\gamma})=\mathbf{0}$, where $\mathbf{w}_{ni}$ is the score (for maximum likelihood estimation) or score-like vector (for GEE estimation) for the ith subject ($1\leq i\leq n$). By a Taylor series expansion, we have

$$\begin{aligned}\sqrt{n}\left(\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}\right) &= \left(-\frac{\partial}{\partial\boldsymbol{\gamma}}\mathbf{w}_n\right)^{-\top}\sqrt{n}\mathbf{w}_n+\mathbf{o}_p(1) \\ &= -H^{-1}\frac{\sqrt{n}}{n}\sum_{i=1}^{n}\mathbf{w}_{ni}+\mathbf{o}_p(1)\end{aligned} \tag{5.99}$$

where $H=E\left(\frac{\partial}{\partial\boldsymbol{\gamma}}\mathbf{w}_{ni}\right)^{\top}$. It follows from (5.98) and (5.99) that (see exercise):

$$\begin{aligned}\sqrt{n}\left(\mathbf{U}_n(\widehat{\boldsymbol{\gamma}})-\boldsymbol{\theta}\right) &= \sqrt{n}\left[\mathbf{U}_n(\widehat{\boldsymbol{\gamma}})-\mathbf{U}_n(\boldsymbol{\gamma})+\mathbf{U}_n(\boldsymbol{\gamma})-\boldsymbol{\theta}\right] \\ &= \sqrt{n}\left(\mathbf{U}_n(\boldsymbol{\gamma})-\boldsymbol{\theta}\right)+\left(\frac{\partial}{\partial\boldsymbol{\gamma}}\mathbf{U}_n(\boldsymbol{\gamma})\right)^{\top}\sqrt{n}\left(\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}\right)+\mathbf{o}_p(1) \\ &= q\frac{\sqrt{n}}{n}\sum_{i=1}^{n}\left(\widetilde{\mathbf{h}}_1\left(\mathbf{y}_i,\mathbf{r}_i,\boldsymbol{\gamma}\right)-CH^{-1}\mathbf{w}_{ni}\right)+\mathbf{o}_p(1)\end{aligned} \tag{5.100}$$

where $C=E\left(\frac{\partial}{\partial\boldsymbol{\gamma}}\widetilde{\mathbf{h}}_1\left(\mathbf{y}_i,\mathbf{r}_i,\boldsymbol{\gamma}\right)\right)^{\top}$. By applying CLT to (5.100), we see that $\widehat{\boldsymbol{\theta}}$ is asymptotically normal with the asymptotic variance given by (see exercise):

$$\begin{aligned}\Sigma_\theta &= 4\left[Var\left(\widetilde{\mathbf{h}}_1\left(\mathbf{y}_i,\boldsymbol{\gamma}\right)\right)+\Phi\right] \\ \Phi &= CH^{-1}Var\left(\mathbf{w}_{ni}\right)H^{-\top}C^{\top}-E\left(\widetilde{\mathbf{h}}_1\left(\mathbf{y}_i,\mathbf{r}_i,\boldsymbol{\gamma}\right)\mathbf{w}_{ni}^{\top}H^{-\top}C^{\top}\right)- \\ &\quad -\left[E\left(\widetilde{\mathbf{h}}_1\left(\mathbf{y}_i,\mathbf{r}_i,\boldsymbol{\gamma}\right)\mathbf{w}_{ni}^{\top}H^{-\top}C^{\top}\right)\right]^{\top}\end{aligned} \tag{5.101}$$