In the presence of missing data, we can readily generalize the approach in Example 1 to accommodate the extra component for the mean response. Let $R_{ij2} = r_{i2}r_{j2}\mathbf{I}_2$ and define the estimating equation as follows:

$$\mathbf{U}_n = \sum_{(i,j)\in C_2^n} \begin{pmatrix} I_2 & 0 \\ 0 & R_{ij2} \end{pmatrix} \left[\begin{pmatrix} \frac{1}{2}(y_{i1}+y_{j1}) \\ \frac{1}{2}(y_{i1}-y_{j1})^2 \\ \frac{1}{2}(y_{i2}+y_{j2}) \\ \frac{1}{2}(y_{i2}-y_{j2})^2 \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \sigma_1^2 \\ \mu_2 \\ \sigma_2^2 \end{pmatrix} \right] = \mathbf{0} \qquad (6.101)$$

As in Example 1, the estimating equation above is well defined. Further, it is readily checked that (6.101) yields the same estimates of σ_1^2 and σ_2^2 as those in Example 1 (see exercise). The estimates of μ_1 and μ_2 are given by (see exercise)

$$\widehat{\mu}_1 = \frac{1}{n} \sum_{(i,j)\in C_2^n} y_{i1}, \quad \widehat{\mu}_2 = \frac{1}{\sum_{i=1}^n r_{i2}} \sum_{i=1}^n r_{i2} y_{i2} \qquad (6.102)$$

As in Example 1, it is readily shown that the estimate $\widehat{\boldsymbol{\theta}}$ obtained from (6.101) with $\widehat{\mu}_t$ and $\widehat{\sigma}_t^2$ given in (6.102) and (6.97) is consistent and asymptotically normal (see exercise).

Example 3 (***Product–moment correlation***). Consider again the product–moment correlation for longitudinal study data discussed in Chapter 5. Let $\mathbf{z}_{it} = (x_{it}, y_{it})^\top$ denote the continuous bivariate outcome from the ith subject at time t from a longitudinal study with n subjects and m assessment times ($1 \le i \le n$, $1 \le t \le m$). Let $\rho_t = Corr(x_{it}, y_{it})$ denote the product–moment correlation between x_{it} and y_{it} and $\boldsymbol{\rho} = (\rho_1, , \ldots, \rho_m)^\top$. We discussed inference about $\boldsymbol{\rho}$ under both MCAR and MAR within the context of multivariate U-statistics in Chapter 5. Alternatively, we can also model $\boldsymbol{\rho}$ using an FRM.

Let

$$f_{kt}(\mathbf{z}_{it}, \mathbf{z}_{jt}) = \begin{cases} \frac{1}{2}(x_{it}-x_{jt})(y_{it}-y_{jt}) \\ \frac{1}{2}(x_{it}-x_{jt})^2 \\ \frac{1}{2}(y_{it}-y_{jt})^2 \end{cases}, \quad h_{kt} = \begin{cases} \sqrt{\sigma_{xt}^2\sigma_{yt}^2}\rho_t & k=1 \\ \sigma_{xt}^2 & k=2 \\ \sigma_{yt}^2 & k=3 \end{cases}$$

$$\mathbf{f}_t = (f_{1t}, f_{2t}, f_{3t})^\top, \quad \mathbf{h}_t = (h_{1t}, h_{2t}, h_{3t})^\top \qquad (6.103)$$

Also, let

$$\boldsymbol{\theta}_t = \left(\rho_t, \sigma_{xt}^2, \sigma_{yt}^2\right)^\top, \quad \boldsymbol{\theta} = \left(\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_m\right)^\top$$

Define the FRM as follows:

$$E\left(\mathbf{f}_t\left(\mathbf{z}_{it}, \mathbf{z}_{jt}\right)\right) = \mathbf{h}_t\left(\boldsymbol{\theta}\right), \quad 1 \leq t \leq m \tag{6.104}$$

In the FRM model above, $\boldsymbol{\rho}$ is part of the model parameter vector $\boldsymbol{\theta}$. In Chapter 5, we first employed multivariate U-statistics to model the covariance $\sigma_{xyt} = Cov\left(x_{it}, y_{it}\right)$ together with the variance components σ_{xt}^2 and σ_{yt}^2 and then obtained inference about $\boldsymbol{\rho}$ by applying the Delta method, as ρ_t is a function of these parameters. In contrast, the FRM in (6.104) directly models $\boldsymbol{\rho}$ as part of its parameters.

Now let

$$\mathbf{f} = \left(\mathbf{f}_1^\top, \ldots, \mathbf{f}_m^\top\right)^\top, \quad \mathbf{h} = \left(\mathbf{h}_1^\top, \ldots, \mathbf{h}_m^\top\right)^\top, \quad \mathbf{z}_i = \left(\mathbf{z}_{i1}^\top, \ldots, \mathbf{z}_{im}^\top\right)^\top$$

Then, by setting $G = D = \frac{\partial}{\partial \boldsymbol{\theta}} \mathbf{h}\left(\boldsymbol{\theta}\right) = \mathbf{I}_{3 \times m}$, the estimating equation for $\boldsymbol{\theta}$ in the absence of missing data is given by

$$\mathbf{U}_n\left(\boldsymbol{\theta}\right) = \sum_{i,j \in C_2^n} D S_{ij} = \sum_{i,j \in C_2^n} \left[\mathbf{f}\left(\mathbf{z}_i, \mathbf{z}_j\right) - \mathbf{h}\right] = \mathbf{0} \tag{6.105}$$

The above UGEE can be solved in closed form to obtain the estimate $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$. In addition, it is readily checked that $\widehat{\rho}_t$ are the Pearson correlation estimates and $\widehat{\sigma}_{xt}^2$ and $\widehat{\sigma}_{yt}^2$ are the sample variances of x_{it} and y_{it}, respectively (see exercise).

When there is missing data, let $r_{it} = 1$ if $\mathbf{z}_{it}$ is observed, that is, if both x_{it} and y_{it} are observed. Also, let

$$R_{ijt} = r_{it} r_{jt} I_3, \quad R_{ij} = \operatorname{diag}_t\left(R_{ijt}\right) = \begin{pmatrix} R_{ij1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & R_{ijm} \end{pmatrix}, \quad 1 \leq i, j \leq n$$

Define the estimating equation in the presence of missing data by

$$U_n\left(\boldsymbol{\theta}\right) = \sum_{(i,j) \in C_2^n} R_{ij}\left(\mathbf{f}\left(\mathbf{z}_i, \mathbf{z}_j\right) - \mathbf{h}\left(\boldsymbol{\theta}\right)\right) = \mathbf{0} \tag{6.106}$$

Like the estimating equations for Examples 1 and 2 discussed above, (6.106) is well defined. It is again readily checked that the estimate obtained as the solution to (6.106) is $\widehat{\boldsymbol{\theta}} = \left(\widehat{\rho}_t, \widehat{\sigma}_{xt}^2, \widehat{\sigma}_{yt}^2\right)^\top$, where $\widehat{\rho}_t$ is the Pearson correlation between x_{it} and y_{it}, while $\widehat{\sigma}_{xt}^2$ and $\widehat{\sigma}_{yt}^2$ are the respective sample variance estimates of x_{it} and y_{it} calculated based on the observed data (see exercises), that is,

$$\widehat{\sigma}_{xt}^2 = \frac{1}{n_t}\sum_{i=1}^n r_{it}\left(x_{it} - \widehat{\mu}_{xt}\right)^2, \quad \widehat{\mu}_{xt} = \frac{1}{n_t}\sum_{i=1}^n r_{it}x_{it}, \quad 1 \leq t \leq m \qquad (6.107)$$

$$\widehat{\sigma}_{yt}^2 = \frac{1}{n_t}\sum_{i=1}^n r_{it}\left(y_{it} - \widehat{\mu}_{yt}\right)^2, \quad \widehat{\mu}_{yt} = \frac{1}{n_t}\sum_{i=1}^n r_{it}y_{it}$$

$$\widehat{\rho}_t = \frac{1}{n_t}\sum_{i=1}^n r_{it}\frac{\left(x_{it} - \widehat{\mu}_{xt}\right)\left(y_{it} - \widehat{\mu}_{yt}\right)}{\sqrt{\widehat{\sigma}_{xt}^2\widehat{\sigma}_{yt}^2}}, \quad n_t = \frac{1}{n}\sum_{i=1}^n r_{it}$$

The consistency and asymptotic normality of the estimate $\widehat{\boldsymbol{\theta}}$ also follows from an argument similar to the one used in Example 2.

Note that in the missing data considerations above, we assumed that both x_{it} and y_{it} are either observed or missing together. As in the case of the multivariate U-statistics approach discussed in Chapter 5 for the product–moment correlation, we can relax this assumption and allow x_{it} and y_{it} to have different missing data patterns. The above considerations are readily generalized to this more general situation (see exercise).

Example 4 (*Linear mixed-effects model*). Consider the FRM defined by (6.86) and (6.87) in Example 9 of Section 6.3.2 for distribution-free inference of the linear mixed-effects model. Let $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{f}_{\mathbf{i}}$, $\mathbf{h}_{\mathbf{i}}$, $S_{\mathbf{i}}$, and $\mathbf{i}$ be defined in (6.88). As discussed in that example, the UGEE in (6.89) defines consistent estimates of the parameter vector of interest $\boldsymbol{\theta}$ under complete data. In the presence of missing data, this estimating equation can be applied to the subset of the data formed by the subjects with complete response y_{it} over all assessment times to provide valid inference about $\boldsymbol{\theta}$. However, as noted earlier, such an approach generally leads to less efficient estimates.

To define an appropriate estimating equation to take advantage of all available data, let $r_{it} = 1$ if y_{it} is observed and 0 if otherwise. We assume no missing data at $t = 1$ so that $r_{i1} \equiv 1$. Also, let

$$\mathbf{r}_i = \left(r_{i1}, r_{i2}, \ldots, r_{im}\right)^\top, \quad \mathbf{r}_{1ij} = \left(r_{i1}r_{j1}, r_{i2}r_{j2}, \ldots, r_{im}r_{jm}\right)^\top \qquad (6.108)$$

$$\mathbf{r}_{2ij} = \left(r_{i1}r_{j1}, r_{i1}r_{j2}, \ldots, r_{i1}r_{jm}, r_{i2}r_{j2}, \ldots, r_{i(m-1)}r_{jm}, r_{im}r_{jm}\right)^\top$$

$$\mathbf{r}_{ij} = \left(\mathbf{r}_{1ij}^\top, \mathbf{r}_{2ij}^\top\right)^\top$$