estimated $\widehat{\boldsymbol{\gamma}}$, we have (see exercise)

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right) = B\sqrt{n}\left[\mathbf{U}_n + \left(\frac{\partial}{\partial\boldsymbol{\gamma}}\mathbf{U}_n\right)^\top(\widehat{\boldsymbol{\gamma}}-\boldsymbol{\gamma})\right] + \mathbf{o}_p(1) \tag{6.120}$$
$$= 2\frac{\sqrt{n}}{n}B\sum_{i=1}^{n}\left(\widetilde{\mathbf{h}}_1(\mathbf{y}_i,\mathbf{r}_i,\boldsymbol{\gamma}) - CH^{-1}\mathbf{w}_{ni}\right) + \mathbf{o}_p(1)$$

where $C = E\left(\frac{\partial}{\partial\boldsymbol{\gamma}}\widetilde{\mathbf{h}}_1(\mathbf{y}_i,\mathbf{r}_i,\boldsymbol{\gamma})\right)^\top$ and $\widetilde{\mathbf{h}}_1(\mathbf{y}_1,\mathbf{r}_1,\boldsymbol{\gamma}) = E(\mathbf{U}_{n12} \mid \mathbf{y}_1,\mathbf{r}_1)$. Note that the coefficient 2 in the last equality of (6.120) is from the asymptotic expansion of $\mathbf{U}_n$ around its projection. By applying CLT to (6.120), we obtain the asymptotic variance of $\widehat{\boldsymbol{\theta}}$ (see exercise):

$$\Sigma_\theta = 4B(\Sigma_U + \Phi)B \tag{6.121}$$
$$\Phi = CH^{-1}Var(\mathbf{w}_{ni})H^{-\top}C^\top - E\left(\widetilde{\mathbf{h}}_1(\mathbf{y}_i,\mathbf{r}_i,\boldsymbol{\gamma})\mathbf{w}_{ni}^\top H^{-\top}C^\top\right) -$$
$$-\left[E\left(\widetilde{\mathbf{h}}_1(\mathbf{y}_i,\mathbf{r}_i,\boldsymbol{\gamma})\mathbf{w}_{ni}^\top H^{-\top}C^\top\right)\right]^\top$$

where Σ_U is defined in (6.116) and Φ accounts for the additional variability due to $\widehat{\boldsymbol{\gamma}}$. A consistent estimate of Φ is readily obtained by substituting consistent estimates for the respective quantities in (6.121) (see exercise).

Note that the above consideration only applies to the special case with $G = D = \frac{\partial}{\partial\boldsymbol{\theta}}\mathbf{h}(\boldsymbol{\theta})$. If G is also parameterized by a vector $\boldsymbol{\alpha}$, the UWGEE estimate $\widehat{\boldsymbol{\theta}}$ from (6.112) is still consistent when $\boldsymbol{\alpha}$ is substituted by some estimate $\widehat{\boldsymbol{\alpha}}$. Further, if $\widehat{\boldsymbol{\alpha}}$ is $\sqrt{n}$-consistent, we can estimate the asymptotic variance of $\widehat{\boldsymbol{\theta}}$ by (6.116) if $\pi_{\mathbf{i}2}$ are known or by (6.121) if $\pi_{\mathbf{i}2}$ are unknown and estimated according to (6.118). In other words, the variability of $\widehat{\boldsymbol{\alpha}}$ does not affect the asymptotic variance of $\widehat{\boldsymbol{\theta}}$. This is readily established by a slight modification of the asymptotic expansion in (6.120) (see exercise). These asymptotic properties parallel those of the WGEE estimates for single-response-based regression models discussed in Chapter 4.

Example 6 (***Linear mixed-effects model***). Consider the FRM for distribution-free inference of the linear mixed-effects model in Example 4 of Section 6.4.1. Under MAR, the estimating equation in (6.109) generally does not provide consistent estimates of $\boldsymbol{\theta}$. As in Example 5 above, a weighted estimating equation must be constructed to ensure valid inference.

As discussed in Example 4, we can use the following modified UGEE to obtain consistent estimates of $\boldsymbol{\theta}$:

$$\mathbf{U}_n(\boldsymbol{\theta}) = \sum_{\mathbf{i}\in C_2^n}\mathbf{U}_{n\mathbf{i}}(\boldsymbol{\theta}) = \sum_{\mathbf{i}\in C_2^n}G_\mathbf{i}R_\mathbf{i}S_\mathbf{i} = \sum_{\mathbf{i}\in C_2^n}G_\mathbf{i}R_\mathbf{i}(\mathbf{f_i}-\mathbf{h_i}) = \mathbf{0} \tag{6.122}$$