By substituting $\frac{1}{n}\frac{\partial l_n}{\partial \boldsymbol{\theta}}$ for $\mathbf{u}_n(\boldsymbol{\theta})$, we have

$$\begin{aligned}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^{\top}\frac{\partial l_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} &= -\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^{\top}\frac{\partial^2 l_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}^{\top}}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)+ \\ &\quad + n\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^{\top} o_p\left(n^{-\frac{1}{2}}\right) \\ &= \left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^{\top} I_n^o(\boldsymbol{\theta})\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)+n\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^{\top} o_p\left(n^{-\frac{1}{2}}\right)\end{aligned} \tag{2.39}$$

Since

$$\begin{aligned} n\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^{\top} o_p\left(n^{-\frac{1}{2}}\right) &= \sqrt{n}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^{\top}\sqrt{n}o_p\left(n^{-\frac{1}{2}}\right) \\ &= O_p(1)\,o_p(1) = o_p(1)\end{aligned} \tag{2.40}$$

it follows from (2.39) and (2.40) that

$$\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^{\top}\frac{\partial l_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^{\top} I_n^o(\boldsymbol{\theta}).\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)+o_p(1) \tag{2.41}$$

Now, expand $l_n\left(\widehat{\boldsymbol{\theta}}\right)$ around $l_n(\boldsymbol{\theta})$ to obtain

$$\begin{aligned} l_n\left(\widehat{\boldsymbol{\theta}}\right)-l_n(\boldsymbol{\theta}) &= \left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^{\top}\frac{\partial l_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}+ \\ &\quad +\frac{1}{2}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^{\top}\frac{\partial^2 l_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}^{\top}}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)+o_p(1) \\ &= \left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^{\top}\frac{\partial l_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}-\frac{1}{2}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^{\top} I_n^o(\boldsymbol{\theta})\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)+o_p(1)\end{aligned} \tag{2.42}$$

By combining (2.41) and (2.42), we have

$$\begin{aligned} -2\ln R(\boldsymbol{\theta}_0) &= 2\left(l_n\left(\widehat{\boldsymbol{\theta}}\right)-l_n(\boldsymbol{\theta})\right) \\ &= \left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)^{\top} I_n^o(\boldsymbol{\theta})\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)+o_p(1)\end{aligned} \tag{2.43}$$

Since

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}-\boldsymbol{\theta}\right)\rightarrow_d N\left(0,\left(I_1^e(\boldsymbol{\theta})\right)^{-1}\right), \quad \frac{1}{n}I_n^o(\boldsymbol{\theta})\rightarrow_p I_1^e(\boldsymbol{\theta}) \tag{2.44}$$

the conclusion follows from (2.43), (2.44), and an application of Slutsky's theorem.

Case 2. Assume $K=(I_l,\mathbf{0})$ with I_l denoting the $l\times l$ identity matrix. Then, the null becomes $H_0:\boldsymbol{\theta}_1=\mathbf{b}$. In this case, the first $l\times 1$ subvector